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BAER'S LEMMA AND FUCHS' PROBLEM 84a

BY

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ABSTRACT. An indecomposable, torsion-free, reduced abelian group A has the properties that (i) each subgroup B of an A-projective group with $S_A(B) = B$ is A-projective and (ii) each subgroup B of a group G with $S_A(G) + B = G$ and G/B A-projective is a direct summand if and only if A is self-small and flat as a left $E(A)$-module, and $E(A)$ is right hereditary. Furthermore, a group-theoretic characterization is given for torsion-free, reduced abelian groups with a right and left Noetherian, hereditary endomorphism ring. This is applied to Fuchs' Problem 84a. Finally, various applications of the results of this paper are given.

1. Introduction. One of the main differences between the theory of torsion-free abelian groups and abelian p-groups is the existence of indecomposable torsion-free abelian groups of arbitrary cardinality. In contrast, the cocyclic groups $\mathbf{Z}(p^n)$ with $n = 1, 2, \ldots, \infty$ are the only indecomposable abelian p-groups. Therefore, it is not surprising that there are only a few results that guarantee that a sequence $0 \to B \to G \to H \to 0$ of torsion-free abelian groups splits. The most important is the following:

BAER'S LEMMA. *Let A be a subgroup of $\mathbf{Q}$, the rational numbers. If B is a subgroup of the abelian group G such that $G/B \cong \oplus_I A$ and $S_A(G) + B = G$, then B is a direct summand of G. Moreover, if $B \subseteq \oplus_J A$ and $S_A(B) = B$, then $B \cong \oplus_K A$.*

Here, we define $S_A(G) = \Sigma\{f(A) \mid f \in \operatorname{Hom}(A,G)\}$ to be the A-socle of G. Naturally, one would like to extend this result to include larger classes of abelian groups from which A can be chosen. To obtain a formulation which is suitable for such a generalization, we introduce the following notations.

If A and G are abelian groups, then G is A-projective if $G \oplus H \cong \oplus_I A$ for some index-set I. It can easily be seen that A-projective groups are not direct sums of copies of A, in general. Consequently, if we want to obtain a generalization of Baer's Lemma, we have to find a class of torsion-free, reduced abelian groups which is as large as possible and satisfies the following conditions:

(I) If U is a subgroup of an A-projective group with $S_A(U) = U$, then U is A-projective.

(II) If B is a subgroup of G such that G/B is A-projective and $S_A(G) + B = G$, then B is a direct summand of G.

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A partial answer was given by Arnold and Lady in [**4**]. They showed that a torsion-free reduced abelian group A of finite rank has a right hereditary endomorphism ring if and only if A satisfies (I) and (II) in the case that G has finite rank and is torsion-free. However the case that A or G have infinite rank was not discussed.

The author showed in [**1** and **3**] that a torsion-free, reduced abelian group satisfies (I) and (II) if A is self-small (i.e. $\mathrm{Hom}(A, -)$ preserves direct sums of copies of A) and flat as an $E(A)$-module, and $E(A)$ is right hereditary. It was shown that this class not only contains the groups described by Arnold and Lady, but also each torsion-free, reduced abelian group with a semiprime, right and left Noetherian, hereditary endomorphism ring regardless of its rank. However, since a generalization of homogeneous separable groups was the goal of [**3**], Baer's Lemma itself was discussed only in a depth necessary for that purpose. The goal of this paper is to discuss Baer's Lemma and related problems.

In its first part, we discuss torsion-free, reduced abelian groups A that satisfy conditions (I) and (II). We show that such an A is flat as an $E(A)$-module and has a right semihereditary endomorphism ring. Moreover, if A is self-small, then $E(A)$ is right hereditary (Corollary 2.3). The final result of §2 will give a necessary and sufficient condition on an indecomposable, torsion-free reduced abelian group to satisfy (I) and (II):

COROLLARY 2.7. *Let A be a torsion-free, reduced, indecomposable abelian group. The following are equivalent*:

(a) *A is self-small and flat as an $E(A)$-module, and $E(A)$ is right hereditary.*

(b) *A satisfies* (I) *and* (II).

The results of this section are related to Problem 84 in [**7**] where Professor Fuchs asks to find criteria that the endomorphism ring of an abelian group A belongs to various classes of rings, for example $E(A)$ is hereditary, $E(A)$ is Noetherian, $E(A)$ is a principal ideal domain. This problem is discussed in the second part of this paper. Interest concentrates on torsion-free, reduced abelian groups A whose endomorphism ring is semiprime, right and left Noetherian, and hereditary. Because of their importance for [**1** and **3**], the question arises whether there are conditions on an abelian group A similar to (I) and (II) that will ensure that $E(A)$ belongs to this class of rings. A positive answer to this question is given in Theorem 5.1 and Lemma 4.1. Due to the length of these results, they will not be stated in this introduction.

Furthermore, as a consequence of the work done so far, we obtain the following answer to Fuchs' Problem 84a:

COROLLARY 5.3. *Let A be an abelian group. The following are equivalent*:

(a) *$E(A)$ is a principal ideal domain.*

(b) *A belongs to one of the following classes of abelian groups*:

(α) *$A \cong \mathbf{Z}(p)$ for some prime p of $\mathbf{Z}$.*

(β) *$A \cong \mathbf{Z}(p^\infty)$ for some prime p of $\mathbf{Z}$.*

(γ) *$A \cong J_p$ for some prime p of $\mathbf{Z}$.*

(δ) *$A \cong \mathbf{Q}$.*

(ε) *A is cotorsion-free* (*i.e.* $\mathbf{Z}(p)$, $\mathbf{Q}$, $J_p \subseteq A$ *for all primes p of* $\mathbf{Z}$) *and* (i) *$E(A)$ is commutative*, (ii) *A is indecomposable, and* (iii) *A satisfies conditions* (Ia) *and* (II).

While it would be nice to replace (i) by a purely group-theoretic condition, it soon becomes apparent that this would be only very hard to achieve. Moreover, it would be far beyond the framework of this paper. Nevertheless, Corollary 5.3 presents by far the most general answer to Problem 84a known to date.

The paper concludes with some applications of the results obtained. An abelian group G is locally A-projective if every finite subset of G is contained in a direct summand of G which is A-projective. In [**3**], various characterizations of locally A-projective groups have been given. In §6, a further one is added:

COROLLARY 6.6. *Let A be a torsion-free reduced abelian group with $E(A)$ right and left Noetherian, hereditary. The following are equivalent for an abelian group G:*

(a) *G is locally A-projective.*

(b) *$S_A(G) = G$ and there is an index-set I such that $G \subseteq \prod_I A$ and $(G + U)/G$ is torsion-free reduced for all A-projective groups U of finite A-rank in $\prod_I A$.*

Here, the A-rank of an A-projective group G is the smallest cardinality of I such that $G \oplus H \cong \oplus_I A$.

2. Baer's Lemma. The goal of this section is to give a rather extensive discussion of torsion-free reduced abelian groups A which satisfy conditions (I) and (II). In [**1** and **3**] the author proved

THEOREM 2.1. *Let A be a torsion-free, reduced abelian group which is self-small and flat as an $E(A)$-module. If $E(A)$ is right hereditary, then A satisfies conditions* (I) *and* (II).

In [**3**], it was shown that the class of groups described in Theorem 2.1 contains all torsion-free reduced abelian groups with a semiprime, right and left Noetherian, hereditary endomorphism ring.

The first step in showing that the converse of Theorem 2.1 holds for indecomposable groups A is the following result whose proof is based on methods developed in [**4**].

PROPOSITION 2.2. *Let A be a torsion-free, reduced abelian group which satisfies conditions* (I) *and* (II). *Then A is a flat $E(A)$-module, and $E(A)$ is right semihereditary.*

PROOF. Let I be a finitely generated right ideal of $E(A)$. Write I as an image of a finitely generated free right $E(A)$-module, say

$$\oplus_n E(A) \xrightarrow{\pi} I \to 0.$$

Tensoring with A induces an epimorphism

$$\left[\oplus_n E(A)\right] \otimes_{E(A)} A \xrightarrow{\pi \otimes \mathrm{id}_A} I \otimes_{E(A)} A \to 0.$$

If $\delta\colon I \otimes_{E(A)} A \to IA$ is defined by $\delta(i \otimes a) = i(a)$, then

$$\delta(\pi \otimes \mathrm{id}_A)\colon \left[\oplus_n E(A)\right] \otimes_{E(A)} A \to IA$$

is an epimorphism. Since $S_A(IA) = IA \subseteq A$, the group IA is A-projective by condition (I). Moreover, (II) implies that $\delta(\pi \otimes \mathrm{id}_A)$ splits. Therefore, the top row of the diagram

$$\begin{array}{ccc} \mathrm{Hom}\left(A, \left[\oplus_n E(A)\right] \otimes_{E(A)} A\right) & \overset{\mathrm{Hom}(A,\delta(\pi\otimes \mathrm{id}_A))}{\rightarrow} & \mathrm{Hom}(A, IA) \rightarrow 0 \\ \uparrow \phi_{\oplus_n E(A)} & & \uparrow j \\ \oplus_n E(A) & \overset{\pi}{\rightarrow} & I \rightarrow 0 \end{array}$$

is splitting exact where $\phi_{\oplus_n E(A)}(x)(a) = x \otimes a$ and j is the evaluation map. Because of

$$\begin{aligned} \left(\mathrm{Hom}(A, \delta(\pi \otimes \mathrm{id}_A))\phi_{\oplus_n E(A)}\right)(x)(a) &= \delta(\pi \otimes \mathrm{id}_A)(x \otimes a) \\ &= \pi(x)(a) = j\pi(x)(a), \end{aligned}$$

the diagram commutes. Thus, j is onto. Furthermore, since j is the evaluation map, it is a monomorphism. IA is isomorphic to a direct summand of $\oplus_n A$, so $I \cong \mathrm{Hom}(A, IA)$ is a projective right $E(A)$-module, and $E(A)$ is right semihereditary.

To show that A is a flat left $E(A)$-module, it suffices to prove that δ is an isomorphism by [**9**, Theorem 3.36]. Consider the diagram:

$$\begin{array}{ccc} \mathrm{Hom}\left(A, (I \otimes_{E(A)} A)\right) & \overset{\mathrm{Hom}(A,\delta)}{\rightarrow} & \mathrm{Hom}(A, IA) \\ \uparrow \phi_I & & \| \\ I & \overset{j}{\rightarrow} & \mathrm{Hom}(A, IA) \end{array}$$

Because of $\mathrm{Hom}(A,\delta)\phi_I(i)(a) = \delta\phi_I(i)(a) = \delta(i \otimes a) = i(a) = j(i)(a)$, it commutes. In the preceding paragraph, we have shown that j is an isomorphism. Since I is a finitely generated, projective right $E(A)$-module, ϕ_I is an isomorphism [**5**]. Hence, $\mathrm{Hom}(A, \delta)$ is also one. Tensoring with A induces a diagram

$$\begin{array}{ccc} \mathrm{Hom}\left(A, (I \otimes_{E(A)} A)\right) \otimes_{E(A)} A & \overset{\mathrm{Hom}(A,\delta)\otimes \mathrm{id}_A}{\underset{\sim}{\rightarrow}} & \mathrm{Hom}(A, IA) \otimes_{E(A)} A \\ \downarrow \theta_{I \otimes_{E(A)} A} & & \downarrow \theta_{IA} \\ I \otimes_{E(A)} A & \overset{\delta}{\rightarrow} & IA \end{array}$$

where $\theta_{IA}(f \otimes a) = f(a)$ and $\theta_{I \otimes_{E(A)} A}(g \otimes a) = g(a)$. Since both groups $I \otimes_{E(A)} A$ and IA are direct summands of a finite number of copies of A, the maps θ_{IA} and $\theta_{I \otimes_{E(A)} A}$ are isomorphisms, and the same holds for δ.

COROLLARY 2.3. *Let A be a self-small, torsion-free, reduced abelian group. The following are equivalent*:

(a) *A satisfies conditions* (I) *and* (II).

(b) *A is a flat left $E(A)$-module, and $E(A)$ is right hereditary.*

PROOF. Let I be any right ideal of $E(A)$. Write I as an epimorphic image of a free right $E(A)$-module, say $\oplus_J E(A) \xrightarrow{\pi} I \to 0$. In order for the argumentation of the proof of Proposition 2.2 to be valid, it is necessary that the map $\phi_{\oplus_J E(A)}$: $\oplus_J E(A) \to \mathrm{Hom}(A, [\oplus_J E(A)] \otimes_{E(A)} A)$ is an isomorphism. But this is guaranteed by the self-smallness of A. Hence I is projective.

Before continuing the discussion, the following technical but useful lemma shall be proved.

LEMMA 2.4. *Let A be a torsion-free, reduced abelian group which satisfies conditions* (I) *and* (II). *If r is a right regular element of $E(A)$ ($rx \neq 0$ for $0 \neq x \in E(A)$), then r is a monomorphism.*

PROOF. Because of (I), $r(A)$ is A-projective. By (II), $A = \ker(r) \oplus C$. Let π: $A \to \ker(r)$ be a projection. Then $r\pi = 0$ implies $\pi = 0$. Therefore, r is a monomorphism.

In order to improve Corollary 2.3, we will introduce a mild restriction on the class of groups from which A can be chosen. Keeping in mind that we are interested in torsion-free abelian groups A which are well behaved with respect to direct sum decompositions, we want to exclude the case that A contains an ascending chain $\{U_i\}_{i \in I}$ of direct summands of infinite length, since this property is more reminiscent of p-groups. The next result describes this property in terms of the endomorphism ring of A:

LEMMA 2.5. *A group A contains an ascending chain $\{U_i\}$ of direct summands of A of infinite length if and only if $E(A)$ contains an infinite set of orthogonal idempotents $\{e_i \mid i < \omega\}$, where ω denotes the first infinite ordinal number.*

PROOF. Suppose A contains such a family $\{U_i\}_{i<\omega}$. Let e_1 be an idempotent in $E(A)$ with $e_1(A) = U_1$. Suppose we have constructed orthogonal idempotents $e_1, \dots, e_n$ with $U_n = (e_1 + \cdots + e_n)(A)$. Write

$$A = U_n \oplus V_n, \quad \text{where } V_n = (1 - e_1 - \cdots - e_n)(A).$$

Then $U_{n+1} = U_n \oplus W_n$, where $W_n = U_{n+1} \cap V_n$. Let e_{n+1} be an idempotent of $E(A)$ with $e_{n+1}(A) = W_n$ and $U_n \subseteq \ker e_{n+1}$. For $i = 1, \dots, n$, we have that for all $x \in A$

$$e_i e_{n+1}(x) \in e_i(V_n) = e_i(1 - e_1 - \cdots - e_n)(A) = (e_i - e_i^2)(A) = 0$$

since $e_{n+1}(x) \in W_n \subseteq V_n = (1 - e_1 - \cdots - e_n)(A)$. Moreover, $e_{n+1}e_i(A) \subseteq e_{n+1}(U_n) = 0$. Thus, $\{e_1, \dots, e_{n+1}\}$ is a set of orthogonal idempotents.

Finally,

$$\begin{aligned} U_{n+1} &= U_n \oplus W_n = U_n \oplus e_{n+1}(A) \\ &= (e_1 + \cdots + e_n)(A) + e_{n+1}(A) = (e_1 + \cdots + e_{n+1})(A). \end{aligned}$$

It remains to show the last equality. Let $a_1, a_2 \in A$. We obtain

$$\begin{aligned} &(e_1 + \cdots + e_n)(a_1) + e_{n+1}(a_2) \\ &\quad = (e_1 + \cdots + e_{n+1})(e_1 + \cdots + e_n)(a_1) + (e_1 + \cdots + e_{n+1})e_{n+1}(a_2) \\ &\quad \in (e_1 + \cdots + e_{n+1})(A). \end{aligned}$$

The family $\{e_i \mid i < \omega\}$ which is obtained this way is an infinite family of orthogonal idempotents.

Conversely, if $\{e_i \mid i < \omega\}$ is an infinite family of orthogonal idempotents of $E(A)$, then $U_n = (e_1 + \cdots + e_n)(A)$ is the required ascending chain of direct summands of A.

The condition that a ring has no infinite sets of orthogonal idempotents is often used in ring-theory while studying hereditary rings. Therefore, it is natural to introduce it at this point. We can now prove

THEOREM 2.6. *Let A be a torsion-free, reduced abelian group such that $E(A)$ contains no infinite family of orthogonal idempotents. The following are equivalent*:

(a) *A is self-small and flat as a left $E(A)$-module, and $E(A)$ is right hereditary.*

(b) (i) *A satisfies conditions* (I) *and* (II).

(ii) *If U is a pure subgroup of A with $R_A(A/U) = 0$, then $A = U \oplus V$, where* $R_A(A/U) = \bigcap\{\ker f \mid f \in \mathrm{Hom}(A/U, A)\}$.

PROOF. (a) $\Rightarrow$ (b): It remains to show part (ii).

Let U be a subgroup of A with $R_A(A/U) = 0$. There is $0 \neq \phi_1 \in E(A)$ with $U \subseteq \ker\phi_1$. By conditions (I) and (II), $A = \ker\phi_1 \oplus U_1$. Either $U = \ker\phi_1$ or there is $0 \neq \psi \in \mathrm{Hom}(\ker\phi_1, A)$ with $\psi(U) = 0$. If we extend ψ to a map ϕ_2 on all of A with $\phi_2(U_1) = 0$, then there is a subgroup U_2 of A with

$$A = \ker\phi_2 \oplus U_2 = (\ker\phi_2 \cap \ker\phi_1) \oplus U_1 \oplus U_2.$$

Moreover, $U \subseteq \ker\phi_2 \cap \ker\phi_1$. If U is not a direct summand of A, then one obtains a decomposition $A = U_1 \oplus \cdots \oplus U_n \oplus V_n$ with $V_n \supseteq U$ and $\mathrm{Hom}(V_n/U, A) \neq 0$. As in the case $n = 2$, $V_n = U_{n+1} \oplus V_{n+1}$ and $V_{n+1} \supseteq U$. By Lemma 2.5, $E(A)$ contains an infinite set of orthogonal idempotents, a contradiction.

(b) $\Rightarrow$ (a): By Proposition 2.2, A is flat as an $E(A)$-module and $E(A)$ is semihereditary. By Corollary 2.3, it suffices to show that A is self-small. Suppose this does not hold. [5, Proposition 1.1] implies that A is the union of an ascending chain of subgroups $\{A_n\}_{n<\omega}$ with $A_n^* = \{f \in E(A) \mid f(A_n) = 0\} \neq 0$. Let $A_n^{**} = \bigcap\{\ker f \mid f \in A_n^*\}$. Then, $A_n \subseteq A_n^{**}$ and $A_n^* = A_n^{***}$. Moreover, $A_n \subseteq A_{n+1}$ implies $A_n^{**} \subseteq A_{n+1}^{**}$. Define ϕ: $A \to \prod_{f \in A_n^*} A_f$, where $A_f \cong A$ by $\phi(a) = (f(a))_{f \in A_n^*}$. Then, $\ker\phi = A_n^{**}$ and $A/A_n^{**} \subseteq \prod_{f \in A_n^*} A_f$. Therefore, $R_A(A/A_n^{**}) = 0$, and A_n^{**} is a direct summand of A.

If $A_n^{**} \subsetneqq A_{n+1}^{**}$ for infinitely many n, then $E(A)$ has an infinite set of orthogonal idempotents by Lemma 2.5, a contradiction. Hence, there is $n_0 < \omega$ such that $A_n^{**} = A_{n_0}^{**}$ for $n_0 < n < \omega$. Then $A = \bigcup_{n \geqslant n_0} A_n^{**} \subseteq A_{n_0}^{**}$. Thus, $A_{n_0}^{**} = A$ and $0 \neq A_{n_0}^* = A_{n_0}^{***} = A^* = 0$. This final contradiction shows that A is self-small.

This result allows us to prove the converse of Theorem 2.1 in the case that A is indecomposable.

COROLLARY 2.7. *Let A be an indecomposable torsion-free, reduced abelian group. The following are equivalent*:

(a) *A is self-small and flat as an $E(A)$-module, and $E(A)$ is right hereditary.*

(b) *A satisfies conditions* (I) *and* (II).

PROOF. In view of Theorem 2.6, it is enough to show that a subgroup U of A with $R_A(A/U) = 0$ is 0 or A. Suppose $U \subsetneq A$. There is $0 \neq \phi \in E(A)$ with $\phi(U) = 0$. By conditions (I) and (II), $A = \ker\phi \oplus V$ with $V \neq 0$. Since A is indecomposable, $\ker\phi = 0$. Hence $U = 0$.

3. The right conditions. In the last section, we have discussed questions related to Baer's Lemma. In the remaining part of this paper, we are concerned with abelian groups A which have a semiprime, right and left Noetherian, hereditary endomorphism ring, which means

(i) $I^2 \neq 0$ for each nonzero two-sided ideal I of $E(A)$,
(ii) each right and left ideal of $E(A)$ is finitely generated, and
(iii) each right and left ideal is projective as an $E(A)$-module.

LEMMA 3.1. *Let A be a torsion-free reduced abelian group which is flat as an $E(A)$-module. $E(A)$ has finite right Goldie dimension iff, for each subgroup $U = \bigoplus_{i\in I} U_i$ of A with $S_A(U) = U$, $U_i = 0$ for all but finitely many $i \in I$.*

PROOF. If $U = \bigoplus_{i\in I} U_i$ satisfies $S_A(U) = U$, then $S_A(U_i) = U_i$. Suppose $n = \dim E(A)$, where $\dim E(A)$ denotes the right Goldie dimension of $E(A)$. If there are $0 \neq U_1, \ldots, U_{n+1} \in \{U_i \mid i \in I\}$, then $\mathrm{Hom}(A, \bigoplus_{i=1}^{n+1} U_i) \cong \bigoplus_{i=1}^{n+1} \mathrm{Hom}(A, U_i)$ is a right ideal of dimension $n+1$ of $E(A)$, a contradiction.

Conversely, let $\bigoplus_{i\in I} J_i$ be a direct sum of infinitely many nonzero right ideals of $E(A)$. Since A is flat, $\bigoplus_{i\in I}(J_i \otimes_{E(A)} A) \cong \bigoplus_{i\in I}(J_iA) \subseteq A$ and $J_iA \neq 0$, a contradiction.

With this, we are able to prove the first step of the characterization.

THEOREM 3.2. *Let A be a torsion-free, reduced abelian group. The following are equivalent*:

(a) *$E(A)$ is a right Noetherian, right hereditary ring, and A is self-small and flat as an $E(A)$-module.*

(b) (i) *A satisfies* (I) *and* (II).

(ii) *If $U = \bigoplus_{i\in I} U_i$ is an A-projective subgroup of A, then almost all U_i's are zero.*

(iii) *If U is a subgroup of A with $R_A(A/U) = 0$, then U is a direct summand of A.*

PROOF. (b) ⇒ (a): Suppose A contains an infinite, strictly ascending chain $\{U_i\}_{i<\omega}$ of direct summands of A. Then $E(A)$ contains an infinite set of orthogonal idempotents and has infinite Goldie dimension. By Lemma 3.1, this contradicts (b). By Theorem 2.6, A is self-small and flat as a left $E(A)$-module. $E(A)$ is right hereditary and has finite right Goldie dimension by Lemma 3.1. Using [**6**, Corollary 2.25], we obtain that $E(A)$ is right Noetherian.

(a) ⇒ (b): (i) and (iii) hold by Theorem 2.6, while (ii) holds because of Lemma 3.1.

As before, we pay particular attention to the case that A is indecomposable.

COROLLARY 3.3. *Let A be an indecomposable, torsion-free reduced abelian group. The following are equivalent*:

(a) *$E(A)$ is a right Noetherian, right hereditary ring without zero-divisors, and A is self-small and flat as an $E(A)$-module.*

(b) (i) *A satisfies conditions* (I) *and* (II).

(ii) *If* $U \cong A$ *is a subgroup of* A, *then* $S_A(V) = 0$ *for all subgroups* V *of* A *with* $U \cap V = 0$.

PROOF. (a) $\Rightarrow$ (b): $V \cap U = 0$ implies $\mathrm{Hom}(A, U) \cap \mathrm{Hom}(A, V) = 0$. Since $E(A)$ has right Goldie dimension 1, this is only possible if $S_A(U) = 0$ or $S_A(V) = 0$.

(b) $\Rightarrow$ (a): By Corollary 2.7, $E(A)$ is a right hereditary ring, and A is self-small and flat as an $E(A)$-module. Since A is indecomposable, $E(A)$ contains no idempotents. Thus, $E(A)$ has no zero-divisors since it is hereditary. In particular, the nilradical $N(E(A))$ is zero.

Let $0 \neq c \in E(A)$. Then c is right regular. By Lemma 2.3, $\ker c = 0$. Suppose there is a right ideal $0 \neq J$ with $cE(A) \cap J = 0$. Then, $c(A) \cap JA = 0$ since A is flat. Because of $c(A) \cong A$, (b)(ii) implies $JA = 0$, a contradiction. Thus $\dim E(A) = 1$. By [**6**, Corollary 8.25], $E(A)$ is right Noetherian.

4. Nonsingular modules. In order to give the characterization of the class of groups we are interested in, we need some information on the structure of finitely generated nonsingular modules. Here, an R-module M is nonsingular if 0 is the only element of M which has an essential annihilator.

We begin our investigation by showing that only the prime case is of interest, where a ring R is prime if for nonzero two-sided ideals I and J of R the ideal IJ is nonzero.

LEMMA 4.1. *Let A be a torsion-free, reduced abelian group whose endomorphism ring is right and left Noetherian and hereditary. Then, there are fully invariant subgroups $A_1, \ldots, A_n$ such that $A = A_1 \oplus \cdots \oplus A_n$ and $E(A_j)$ is a prime ring.*

PROOF. By [**6**, Theorem 8.21], $E(A) = R_1 \times \cdots \times R_n \times S$, where the R_i's are prime, and S is Artinian. This induces a decomposition of A in fully invariant subgroups, say $A = A_1 \oplus \cdots \oplus A_n \oplus B$, with $E(A_i) = R_i$ and $E(B) = S$. By [**7**, Theorem 111.3], B is the direct sum of a torsion-free divisible group and a finite group. However, A is torsion-free, reduced. Thus, $B = 0$, and the lemma follows.

We say that a ring R satisfies the restricted right minimum condition if R/I is Artinian for each essential right ideal I of R. [**6**, Theorem 8.21] shows that right and left Noetherian, hereditary rings satisfy the restricted right minimum condition.

THEOREM 4.2. *Let R be a semiprime, right Noetherian, right hereditary ring with restricted right minimum condition whose additive group R^+ is reduced and torsion-free. The following are equivalent for a finitely generated right R-module M:*

(a) *M is nonsingular.*

(b) *The additive group M^+ is torsion-free and reduced.*

PROOF. Let M be nonsingular, and choose a prime p of $\mathbf{Z}$. Since pR is an essential right ideal of R, there is no element $0 \neq m \in M$, with $\mathrm{ann}(m) = \{r \in R \mid mr = 0\} \supseteq pR$. Hence, for all $0 \neq m \in M$, $mp \neq 0$ and $M^+$is torsion-free.

If U is the largest $\mathbf{Z}$-divisible subgroup of M^+, then U is an R-submodule of M.

To show this, let $u \in U$ and $0 \neq n \in \mathbf{Z}$. For all $r \in R$, $ur = (u_n n)r = (u_n r)n$, where $u = u_n n$. Hence $ur \in \bigcap_{0 \neq n \in \mathbf{Z}} nM^+ = U$ since M^+ is torsion-free.

Suppose $U \neq 0$. Pick $0 \neq u \in U$, and write $uR \cong R/\mathrm{ann}(u)$. Since M is nonsingular, $\mathrm{ann}(u)$ is not an essential right ideal of R. Let I be a nonzero right ideal of R with $I \cap \mathrm{ann}(u) = 0$. Then, I is a finitely generated, projective R-module since R is right hereditary. Let U' be the submodule of uR corresponding to $(I + \mathrm{ann}(u))/\mathrm{ann}(u)$.

If V is the $\mathbf{Z}$-purification of U', then $U' \subseteq V \subseteq U$. Moreover, V is an R-submodule of M, since for $v \in V$ and $r \in R$, there is $0 \neq m \in \mathbf{Z}$ with $vm \in U'$. Then $(vr)m = (vm)r \in U'$, and so $vr \in V$. Since R is right Noetherian, and M is finitely generated, V is a finitely generated R-module and V/U' is $\mathbf{Z}$-torsion. Hence there is $0 \neq m \in \mathbf{Z}$ with $V \cong Vm \subseteq U'$. Since R is right hereditary, V is projective. In particular, V^+ is reduced.

On the other hand, V^+ is a pure subgroup of the divisible group U^+. Thus V^+ is divisible, a contradiction. Hence, $V = 0$ and the same holds for U.

Conversely, suppose M^+ is torsion-free, reduced. If there is $0 \neq m \in M$ such that $\mathrm{ann}(m)$ is an essential right ideal of R, then $W = mR \cong R/\mathrm{ann}(m)$ is an Artinian submodule of M by the restricted minimum condition. Let $W = W(n!)$. Since $W_n \supseteq W_{n+1}$, there is $n_0 < \omega$ such that $W_n = W_{n_0}$ for all $n_0 \leqslant n < \omega$. Hence, $W_{n_0} = W_{pn_0} \subseteq Wn_0!p \subseteq W_{n_0}p$ for all primes p. Since M^+ is reduced, $W_{n_0} = 0$. However, M^+ also is torsion-free. Thus, $W = 0$, a contradiction.

LEMMA 4.3. *Let R be a ring such that R^+ is torsion-free, reduced.*

(a) *If M is a nonsingular R-module, then M^+ is torsion-free.*

(b) *If M is an R-module with M^+ torsion-free, and A is a torsion-free abelian group with $R = E(A)$ such that A is flat as an $E(A)$-module, then $M \otimes_R A$ is torsion-free.*

PROOF. (a) is clear since pR is an essential right ideal of R for all primes p of $\mathbf{Z}$.

(b) Since M^+ is torsion-free, multiplication by a prime p of $\mathbf{Z}$ induces a monomorphism $0 \to M \overset{p'}{\to} M$. By the flatness of A,

$$0 \to M \otimes_R A \overset{(p')\otimes \mathrm{id}_A}{\longrightarrow} M \otimes_R A$$

is a monomorphism with $((P') \otimes \mathrm{id}_A)(m \otimes a) = (mp) \otimes a = (m \otimes a)p$. Thus, $M \otimes_R A$ is torsion-free.

LEMMA 4.4. *Let R be a semiprime, right Noetherian, right hereditary ring with restricted right minimum condition whose additive group R^+ is torsion-free reduced. Suppose A is an abelian group with $R = E(A)$ such that A is flat as an $E(A)$-module and satisfies* (I) *and* (II). *If in addition, $(\oplus_n A)/P_1 = D \oplus P$ with D torsion-free divisible and P A-projective holds for each pure subgroup P_1 of $\oplus_n A$ with $S_A(P_1) = P_1$ and all $n < \omega$, then each finitely generated, nonsingular right R-module is projective.*

PROOF. Let M be a finitely generated, nonsingular right R-module. It contains an essential submodule U which is the direct sum of finitely many, uniform cyclic submodules, say $U = U_1 \oplus \cdots \oplus U_n$. Since U_i is nonsingular, there is a nonzero, finitely generated projective submodule I_i of U_i. Let $\bar{P} = \oplus_{i=1}^n I_i$, a finitely generated, essential, projective submodule of M, and let P be its $\mathbf{Z}$-purification in M. Then P is a submodule of M and, therefore, finitely generated. Also, $\overline{P}/P$ is $\mathbf{Z}$-torsion. Observe that M^+ is torsion-free by Theorem 4.2. Hence, there is $0 \neq m \in \mathbf{Z}$ with $P \cong Pm \subseteq \overline{P}$ and P is projective.

Since R is right hereditary, there are finitely generated projective modules $F_1 \subseteq F_2$ with $M = F_2/F_1$. Then $0 \to F_1 \otimes_R A \to F_2 \otimes_R A \to M \otimes_R A \to 0$ is pure-exact by Lemma 4.3. We can choose F_2 to be free. So, $M \otimes_R A = D \oplus C$ with D divisible and C A-projective. By conditions (I) and (II), C is isomorphic to a direct summand of $F_2 \otimes_R A$ and $\mathrm{Hom}(A,C)$ is a finitely generated, projective right $E(A)$-module.

Moreover, $P \otimes_R A \subseteq M \otimes_R A$ and

$$\begin{aligned} D/[(P \otimes_R A) \cap D] &\cong [D + (P \otimes_R A)]/(P \otimes_R A) \\ &\subseteq (M \otimes_R A)/(P \otimes_R A) \cong (M/P) \otimes_R A \end{aligned}$$

is torsion-free by Lemma 4.3, since $(M/P)^+$ is torsion-free, and A is flat. Thus, $[D \cap (P \otimes_R A)]$ is a pure subgroup of D. Consequently, $D \cap (P \otimes_R A)$ is divisible. On the other hand, it is contained in $P \otimes_R A$, a direct summand of a reduced group $\oplus A$. So, $D \cap (P \otimes_R A) = 0$. Therefore, we can choose the complimentary summand C for D in such a way that it contains $P \otimes_R A$.

Let π: $\mathrm{Hom}(A, M \otimes_R A) \to \mathrm{Hom}(A,C)$ be the canonical projection with $\ker \pi = \mathrm{Hom}(A, D)$. Define ϕ_M: $M \to \mathrm{Hom}(A, M \otimes_R A)$ by $\phi_M(m)(a) = m \otimes a$. There exists a commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\phi_M} & \mathrm{Hom}(A, M \otimes_R A) \\ \cup\!| & & \cup\!| \\ P & \xrightarrow{\phi_P} & \mathrm{Hom}(A, P \otimes_R A) \end{array}$$

The bottom map is an isomorphism since P is a finitely generated projective. Moreover, $\mathrm{Hom}(A, P \otimes_R A)$ is contained in $\mathrm{Hom}(A,C)$. Thus, $\pi\phi_M|_P$: $P \to \mathrm{Hom}(A,C)$ is a monomorphism, since $0 = \pi\phi_m(p) = \pi\phi_p(p)$ implies $\phi_p(p)$ is an element of $\mathrm{Hom}(A, P \otimes_R A) \cap \ker \pi \subseteq \mathrm{Hom}(A,C) \cap \mathrm{Hom}(A, D) = 0$. Since ϕ_p is a monomorphism, $p = 0$.

Let $m \in \ker \pi\phi_M$. Then $m + P \in M/P$ and $\mathrm{ann}(m + P)$ is an essential right ideal of R. Otherwise, M/P would contain a projective submodule $N \neq 0$. Then $M \supseteq P \oplus \overline{N}$ where $(\overline{N} \oplus P)/P = N$ and

$$\infty > \dim M \geqslant \dim(P \oplus \overline{N}) > \dim(P) = \dim(M),$$

since $N \neq 0$. Consequently, P is not essential in M. On the other hand, P contains $\overline{P}$, and $\overline{P}$ is essential, a contradiction.

Since R is a semiprime, right Noetherian ring, there is a regular element $c \in R$ with $mc \in P$. Thus

$$0 = \pi\phi_m(m)c = \pi\phi_m(mc) = \pi\phi_p(mc).$$

Hence $mc = 0$ and $c \in \mathrm{ann}(m)$. Since R is semiprime, Noetherian, $\mathrm{ann}(m)$ is essential. Hence $m = 0$, and M is isomorphic to a submodule of the projective module $\mathrm{Hom}(A,C)$. Since R is right hereditary, M is projective.

The next step is to show that rings R like the ones in Lemma 4.5 have finite left Goldie dimension.

LEMMA 4.5. *Let R be a semiprime, right Noetherian, right hereditary ring with restricted right minimum condition. If each finitely generated, nonsingular right R-module is projective, then R has finite left Goldie dimension.*

PROOF. By [**8**, Theorems 5.18, 2.11] the right quotient ring Q of R is flat as a right R-module, the multiplication map $Q \otimes_R Q \to Q$ is an isomorphism, and ${}_RR$ is essential in ${}_RQ$.

Since R is left semihereditary, it is nonsingular as a left R-module. By [**8**, Proposition 2.27], Q is a left quotient ring of R. By [**8**, Proposition 2.11] ${}_RQ$ is the injective hull of ${}_RR$. Using [**8**, Theorem 3.17], ${}_RR$ is finite dimensional.

THEOREM 4.6. *Let R be a semiprime, right Noetherian, right hereditary ring. The following are equivalent*:

(a) *R is left Noetherian.*

(b) (i) *R satisfies the restricted minimum condition.*

(ii) *Each finitely generated, nonsingular right R-module is projective.*

PROOF. (b) $\Rightarrow$ (a): Let I be an essential left ideal of R. Since R is left semihereditary and has finite left Goldie dimension, $I \supseteq J$ where J is a finitely generated, essential left ideal of R. Suppose I is not finitely generated. There is an ascending chain $J = J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_n \subsetneq \cdots \subsetneq I \subseteq R$ of finitely generated left ideals of R which are projective.

Let $J_i^* = \{q \in Q \mid J_i q \subseteq R\}$, where Q is the right quotient ring of R. If K is an essential left ideal of R, then R/K is singular, which means that $\operatorname{Hom}_R(R/K, M) = 0$ for each nonsingular R-module M, and

$$0 = \operatorname{Hom}_R(R/K, Q) \to \operatorname{Hom}_R(R, Q) \to \operatorname{Hom}_R(K, Q) \to 0$$

since Q is injective. Thus, each $f \in \operatorname{Hom}_R(J_i, Q)$ is right multiplication by some $q \in Q$ and $\operatorname{Hom}(J_i, R)$ is isomorphic to J_i^*. Therefore, J_i^* is projective.

Since R is a semiprime left Goldie-ring, J can be chosen to be of the form Rc with $c \in R$ regular. Then,

$$R \cong J^* \supseteq \cdots \supseteq J_{n+1}^* \supseteq J_{n+2}^* \supseteq \cdots \supseteq R^* \cong R.$$

By the restricted right minimum condition, J^*/R^* is Artinian since $J^*/R^* = R/X$ for some essential right ideal X of R. Hence, $J_i^* = J_{i+1}^*$ for almost all i. The part (b) $\Rightarrow$ (a) is proven if we can show $J_i = J_i^{**}$.

Since J_i is finitely generated projective, there exist a dual basis $\{q_1, \ldots, q_n\} \subseteq J_i^*$ and $x_1, \ldots, x_n \in J_i$ such that $y = \sum_{i=1}^n q_i(y) x_i$ for all $y \in J_i$. Since $q_i(y) = yq_i$ with $q_i \in Q$, $1 - \sum_{i=1}^n q_i x_i$ is annihilated by the essential left ideal J_i of R. But ${}_RQ$ is nonsingular, so $1 = \sum_{i=1}^n q_i x_i$.

Let $k \in J_i^{**} = \{k \in Q \mid kJ_i^* \subseteq R\}$. Then $k = \sum_{i=1}^n k(q_i x_i) = \sum_{i=1}^n (kq_i) x_i \in J_i$, hence $J_i^{**} \subseteq J_i$. On the other hand, $J_i \subseteq J_i^{**}$. Thus, $J_i = J_i^{**}$, a contradiction. Therefore, R is left Noetherian.

(a) ⇒ (b): By [**6**, Theorem 8.2], R satisfies the restricted minimum condition, while (ii) follows immediately from [**8**, Theorems 3.10, 5.18].

5. The characterization. In this section, we are using the previous results to give necessary and sufficient conditions on a group A that $E(A)$ is a prime, right and left Noetherian, hereditary ring. In view of Lemma 4.1, this solves the second problem stated in the introduction.

THEOREM 5.1. *Let A be a torsion-free, reduced abelian group. The following are equivalent*:

(a) *$E(A)$ is a prime, right and left Noetherian, hereditary ring.*

(b) *A satisfies the following conditions*:

(i) *$A = A_1 \oplus \cdots \oplus A_n$ with A_i indecomposable.*

(ii) *A satisfies conditions* (I) *and* (II).

(iii) *Each A-projective subgroup U of A is contained in a direct summand V of A such that there is a chain $U = V_0 \subseteq \cdots \subseteq V_n = V$ of A-projective subgroups of A with the property that there is no A-projective subgroup W of A with $V_{i-1} \subsetneq W \subsetneq V_i$.*

(iv) *If P is a pure subgroup of $\oplus_n A$ with $S_A(P) = P$, then $(\oplus_n A)/P = D \oplus P_1$, where D is divisible and P_1 is A-projective.*

(v) *If W is a subgroup of A with $R_A(A/W) = 0$, then $A = W \oplus W_1$.*

(vi) *If $U \cong A$ is an A-projective subgroup of A and $U \cap V = 0$ for some subgroup V of A, then $S_A(V) = 0$.*

(vii) *If W is a fully invariant subgroup of A which is contained in a direct summand $0 \neq U \neq A$ of A, then $S_A(W) = 0$.*

PROOF. (b) ⇒ (a): Because of (ii), $E(A)$ is a right semihereditary ring, and A is a flat left $E(A)$-module. Suppose there are $0 \neq f, g \in E(A)$ with $fE(A)g = 0$. Since $E(A)g(A)$ is a nonzero fully invariant subgroup of A with $S_A(E(A)g(A)) = E(A)g(A) \neq 0$ and $\ker(f)$ is a direct summand of A because of conditions (I) and (II), we have a contradiction to $f \neq 0$. Thus, $E(A)$ is a prime ring.

Let c be any right regular element of $E(A)$. By Lemma 2.4, c is a monomorphism. If J is an right ideal of $E(A)$ with $cE(A) \cap J = 0$, then $c(A) \cap JA = 0$ since A is flat. By (vi), $J = 0$. [**6**, Lemma 8.12] implies that $E(A)/N(E(A))$ has finite right Goldie dimension since (i) guarantees $\mathrm{id}_A = e_1 + \cdots + e_n$ with e_i a primitive idempotent of $E(A)$. $E(A)$ is a prime ring, and so $N(E(A)) = 0$. Thus, $E(A)$ has finite right Goldie dimension. In particular, it contains no infinite set of orthogonal idempotents. Because of (ii) and (v), it is possible to apply Theorem 2.5 to show that $E(A)$ is right hereditary and A is self-small. But a right hereditary ring of finite right Goldie dimension is right Noetherian [**6**, Corollary 8.25].

So far, we have shown that $E(A)$ is a prime, right Noetherian, right hereditary ring. In view of the results of §4, we show the restricted right minimum condition for $E(A)$.

Let J be an essential right ideal of $E(A)$. There is a regular element $c \in E(A)$ with $c \in J$. Since $E(A)/J$ is an epimorphic image of $E(A)/cE(A)$, it suffices to show that the latter is an Artinian $E(A)$-module. By (iii), there are A-projective groups

$$c(A) = V_0 \subseteq \cdots \subseteq V_n = V$$

in A with $A = V \oplus U$, and there is no A-projective group properly contained between V_{i-1} and V_i for all i. Since $c(A) \cong A$ and $c(A) \cap U = 0$, (vi) implies $S_A(U) = 0$. But U is A-projective. Therefore, $U = S_A(U) = 0$. We obtain an ascending chain

$$\begin{aligned} cE(A) &\subseteq \mathrm{Hom}(A, c(A)) = \mathrm{Hom}(A, V_0) \\ &\subseteq \mathrm{Hom}(A, V_1) \subseteq \cdots \subseteq \mathrm{Hom}(A, V_n) = E(A) \end{aligned}$$

of right ideals of $E(A)$.

If $f \in \mathrm{Hom}(A, c(A))$, then write $f(a) = c(a')$. Since c is a monomorphism, $g(a) = a'$ defines an element of $E(A)$ with $f = cg$. Therefore,

$$cE(A) = \mathrm{Hom}(A, c(A)).$$

Suppose there is a right ideal I of $E(A)$ with $\mathrm{Hom}(A, V_{i-1}) \subsetneq I \subsetneq \mathrm{Hom}(A, V_i)$ for some i. If θ_A: $E(A) \otimes_{E(A)} A \to A$ is defined by $\theta(r \otimes a) = r(a)$, we have $V_{i-1} \subseteq \theta_A(I \otimes_{E(A)} A) = IA \subseteq V_i$. IA is A-projective by (ii), so either $V_{i-1} = IA$ or $V_i = IA$. In the first case, we obtain a diagram:

$$\begin{array}{ccc} \mathrm{Hom}(A, V_{i-1}) & & \subsetneq I \\ \wr\downarrow & & \wr\downarrow \\ \mathrm{Hom}\big(A, \mathrm{Hom}(A, V_{i-1}) \otimes_{E(A)} A\big) & = & \mathrm{Hom}(A, IA) \end{array}$$

The vertical maps are isomorphisms, since both $\mathrm{Hom}(A, V_{i-1})$ and I are finitely generated projective. The resulting contradiction shows that

$$\mathrm{Hom}(A, V_i)/\mathrm{Hom}(A, V_{i-1})$$

is a simple $E(A)$-module. Therefore, $E(A)/cE(A)$ has a composition series. In particular, it is Artinian.

Combining (iv) and Corollary 4.3 implies that every finitely generated, nonsingular, right $E(A)$-module is projective. By Theorem 4.5, $E(A)$ is left Noetherian.

(a) $\Rightarrow$ (b): By Theorems 2.6 and 3.2, conditions (i), (ii) and (v) are satisfied. To show (iii), let W be an A-projective subgroup of A. Then $E(A) = X \oplus Y$ where X and Y are right ideals of $E(A)$ such that $\mathrm{Hom}(A, W)$ is an essential submodule of X, and $E(A)/X$ is nonsingular. Then $A = XA \oplus YA$ with $W \subseteq XA$. Let $V = XA$, $U = YA$. Since $E(A)$ satisfies the restricted minimum condition, $X/\mathrm{Hom}(A, W)$ is an Artinian, finitely generated module over a Noetherian ring. Hence there are right ideals $X_0 = \mathrm{Hom}(A, W) \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X$ of $E(A)$ such that X_i/X_{i-1} is simple. Let $V_i = X_i A$ and suppose there is an A-projective group P with $V_{i-1} \subsetneq P \subsetneq V_i$ for some $i \in \{1, \dots, n\}$. Then $X_{i-1} \subsetneq \mathrm{Hom}(A, P) \subsetneq \mathrm{Hom}(A, V_i)$, a contradiction. Thus, (iii) holds.

To show (iv), choose a submodule X of $\mathrm{Hom}(A, \oplus_n A)$ such that $\mathrm{Hom}(A, P)$ is essential in X and $\mathrm{Hom}(A, \oplus_n A)/X$ is nonsingular, where $0 \to P \to \oplus_n A$ is the given pure embedding of A-projective groups. Since $E(A)$ is a prime, right and left

Noetherian, and hereditary ring, finitely generated, nonsingular modules are projective [6]. Thus, $\operatorname{Hom}(A, \oplus_n A) = X \oplus Y$, and $\oplus_n A = XA \oplus YA$ with $XA \supseteq P$. Since Y is projective, YA is A-projective and it suffices to show that XA/P is divisible:

$$XA/P \cong \left(X \otimes_{E(A)} A\right)/\left(\operatorname{Hom}(A, P) \otimes_{E(A)} A\right) \cong (X/\operatorname{Hom}(A, P)) \otimes_{E(A)} A,$$

$X/\operatorname{Hom}(A, P)$ is a finitely generated, singular $E(A)$-module. Because of the restricted minimum condition, it is Artinian. Finally, since P is pure in $\oplus_n A$, $(X/\operatorname{Hom}(A, P))^+$ is torsion-free. Consequently, $(X/\operatorname{Hom}(A, P))^+$ is divisible.

Let $U \cong A$ be a subgroup of A and V an A-projective subgroup of A with $U \cap V = 0$. Then, $\operatorname{Hom}(A, U) \cap \operatorname{Hom}(A, V) = 0$. Since $U \cong A$, $\operatorname{Hom}(A, U) \cong E(A)$ is an essential right ideal of $E(A)$. Thus, $\operatorname{Hom}(A, V) = 0$. But this is only possible if $V = 0$ since V is A-projective.

It is left to show (vii). For this write $A = U \oplus V$ with $U, V \neq 0$, and suppose there is a fully invariant subgroup W of A contained in U. Let $f \in \operatorname{Hom}(A, W)$. Then $E(A)f(A) \subseteq W \subseteq U$. If $\pi\colon A \to V$ is a projection with $\ker(\pi) = U$, then $\pi E(A)f = 0$. Since $\pi \neq 0$ and $E(A)$ is prime, $f = 0$.

Considering the indecomposable case, we obtain a simplification of the last result:

Corollary 5.2. *For a torsion-free, reduced abelian group A, the following are equivalent*:

(a) *$E(A)$ is a right and left Noetherian, hereditary ring without zero divisors.*

(b) *A satisfies the following conditions*:

(i) *A is indecomposable.*

(ii) *A satisfies conditions* (I) *and* (II).

(iii) *If $U \cong A$ is a subgroup of A and $V \subseteq A$ with $U \cap V = 0$, then $S_A(V) = 0$.*

(iv) *If $U \cong A$ is a subgroup of A, then there are A-projective subgroups U_i in A such that $U = U_0 \subseteq \cdots \subseteq U_n = A$ and there is no A-projective V in A with $U_{i-1} \subsetneq V \subsetneq U_i$.*

(v) *If P is a pure, A-projective subgroup of $\oplus_n A$, then $(\oplus_n A)/P = D \oplus C$ with D divisible and C A-projective.*

Proof. (a) $\Rightarrow$ (b) is obvious in view of Theorem 5.1.

(b) $\Rightarrow$ (a): By Corollary 3.3, $E(A)$ is a right Noetherian, right hereditary ring without zero divisors. Moreover, A is self-small and flat as an $E(A)$-module. As in the proof of Theorem 5.1, (iv) implies that $E(A)$ has the restricted right minimum condition. This and (v) guarantee that $E(A)$ is left Noetherian.

A slight modification of condition (I) allows us to give an answer to Fuchs' Problem 84a:

(Ia) If B is a subgroup of $\oplus_I A$ with $S_A(B) = B$, then $B \cong \oplus_J A$.

Corollary 5.3. *Let A be an abelian group. The following are equivalent*:

(a) *$E(A)$ is a principal ideal domain.*

(b) *A belongs to one of the following classes of abelian groups*:

(α) *$A \cong \mathbf{Z}(p)$ for some prime p of $\mathbf{Z}$.*

(β) *$A \cong \mathbf{Z}(p^\infty)$ for some prime p of $\mathbf{Z}$.*

(γ) *$A \cong J_p$ for some prime p of* $\mathbf{Z}$.

(δ) *$A \cong$* $\mathbf{Q}$.

(ε) *A is cotorsion-free (i.e.* $\mathbf{Z}(p)$, $\mathbf{Q}$, *$J_p \subseteq A$ for all primes p of* $\mathbf{Z}$) *and* (i) *$E(A)$ is commutative*, (ii) *A is indecomposable, and* (iii) *A satisfies conditions* (Ia) *and* (II).

PROOF. We only have to show that a group in (ε) has a principal ideal domain as an endomorphism ring iff (i)–(iii) are satisfied. If $E(A)$ is a principal ideal domain, then we have to show condition (Ia). Let $B \subseteq \oplus_I A$ and $S_A(B) = B$. Then, $\mathrm{Hom}(A, B) \subseteq \oplus_I E(A)$, and there is an index-set J with $\mathrm{Hom}(A, B) \cong \oplus_J E(A)$. Thus,

$$B \cong \mathrm{Hom}(A, B) \otimes_{E(A)} A \cong \oplus_J \big(E(A) \otimes_{E(A)} A\big) = \oplus_J A.$$

Conversely, $E(A)$ is a right hereditary ring without zero-divisors by Corollary 3.3. To show that it is a principal ideal domain it suffices to prove that $E(A) \cong I$ for each nonzero right ideal I of $E(A)$. But $IA = I \otimes_{E(A)} A$ is an A-projective group. By condition (Ia), $IA = \oplus_J A$. Moreover, $I \cong \mathrm{Hom}(A, IA) \cong \oplus_J E(A)$ since A is self-small. Comparing the Goldie dimensions of I and $\oplus_J E(A)$ shows $|J| = 1$ and $I \cong E(A)$.

6. Applications. This paper concludes with some applications of the result of the previous sections to the generalizations of homogeneous separable groups. We begin with the introduction of supporting subgroups.

DEFINITION 6.1. Let A and G be abelian groups. A subgroup U of G with $S_A(U) = U$ is an A-subsocle of G. An A-subsocle U of G supports a subgroup W of G if $S_A(W) = U$.

Naturally, the question arises which A-subsocles support pure subgroups. The following result gives a partial answer in the case that $E(A)$ is right and left Noetherian.

PROPOSITION 6.2. *Let A be a torsion-free reduced abelian group whose endomorphism ring is right and left Noetherian. If U is an A-subsocle of a torsion-free abelian group G with $R_A(G) = 0$, then U supports a pure subgroup of G if and only if U is pure in $S_A(G)$.*

PROOF. Let $U = S_A(H)$ for some pure subgroup H of G. Let $g \in S_A(G)$ such that $ng \in U$ for some $0 \neq n \in \mathbf{Z}$.

There are $f_1, \ldots, f_n \in \mathrm{Hom}(A, G)$ with $g \in \sum_{i=1}^n f_i(A)$. If $M = \mathrm{Hom}(A, H)$ and $N = \sum_{i=1}^n f_i E(A)$, then $(N + M)/M$ is a finitely generated $E(A)$-module. Let V be the $\mathbf{Z}$-torsion-subgroup of $(M + N)/M$. V is an $E(A)$-submodule, which is finitely generated. Thus, V is $\mathbf{Z}$-bounded. Write $V = W/M$. For some $0 \neq m \in \mathbf{Z}$, $mW \subseteq M$.

If θ_G: $\mathrm{Hom}(A, G) \otimes_{E(A)} A \to G$ is defined by $\theta_G(f \otimes a) = f(a)$, then

$$U = \theta_G\big(M \otimes_{E(A)} A\big) \subseteq \theta_G\big(W \otimes_{E(A)} A\big) \subseteq U + \sum_{i=1}^{n} f(A)$$

$$= \theta_G\big(N \otimes_{E(A)} A\big) + \theta_G\big(M \otimes_{E(A)} A\big).$$

Moreover,

$$\theta_G((N \otimes_{E(A)} A) + (M \otimes_{E(A)} A))/\theta_G(W \otimes_{E(A)} A) \cong ((N+M)/W) \otimes_{E(A)} A \cong (((N+M)/M)/V) \otimes_{E(A)} A$$

is torsion-free by Lemma 4.3 and because of $R_A(G) = 0$ [**1**, Lemma 6.2].

In addition, $mW \subseteq M$ implies $m\theta_G(W \otimes_{E(A)} A)$ is contained in $\theta_G(M \otimes_{E(A)} A)$. Hence,

$$\theta_G(W \otimes_{E(A)} A)/U = \text{Torsion}\left(\left(U + \sum_{i=1}^{n} f_i(A)\right)/U\right)$$

is bounded by m.

Therefore, $g \in \theta_G(W \otimes_{E(A)} A)$ and

$$m\theta_G(W \otimes_{E(A)} A) \subseteq U \cap mG \subseteq H \cap mG = mH.$$

Thus, $\theta_G(W \otimes_{E(A)} A) \subseteq H$ since G is torsion-free. On the other hand,

$$\theta_G(W \otimes_{E(A)} A) \subseteq S_A(H) = U.$$

Thus $g \in U$ and U is pure in $S_A(G)$.

Conversely, let $U = S_A(U)$ be a pure subgroup of $S_A(G)$ and let V be its purification in G. Then $U = S_A(U) \subseteq S_A(V)$. If $v \in S_A(V)$, then there is $0 \neq n \in \mathbf{Z}$ with $nv \in U \cap nS_A(V) \subseteq U \cap nS_A(G) = nU$. Thus, $v \in U$ and $U = S_A(V)$.

In [**3**], we were concerned with locally A-projective groups G which are exactly the groups G with the property that every finite subset of G is contained in an A-projective direct summand of G.

If we call a subgroup H of an abelian group G with $S_A(G) = G$ almost $\{A\}_*$-pure in G if $S_A(H) = H$ and H is a direct summand of $H + f(A)$ for all $f \in \text{Hom}(A, G)$, then the main results of [**3**] can be summarized as follows:

THEOREM 6.3. *Let A be a torsion-free, reduced abelian group with a semiprime, right and left Noetherian, hereditary endomorphism ring. The following are equivalent for an abelian group G.*

(a) *G is locally A-projective.*

(b) *$G = S_A(G)$ and there is an index-set I such that G is almost $\{A\}_*$-pure in $S_A(\prod_I A)$.*

The results of this paper allow us to add a further, much simpler equivalent statement to Theorem 6.3. Firstly, some preliminary results are needed:

LEMMA 6.4. *Let A be a torsion-free, reduced abelian group with a right and left Noetherian, hereditary endomorphism ring. If G is an abelian group with $S_A(G) = G$ and $R_A(G) = 0$, then a subsocle U of G with $U \cong (\oplus_n A)/V$ is A-projective of finite A-rank, i.e. $U \oplus V = \oplus_m A$ for some $m < \omega$.*

PROOF. See [**3**, Lemma 4.3].

Theorem 6.5. *Let A be a torsion-free reduced abelian group whose endomorphism ring is right and left Noetherian and hereditary. If G is an abelian group with $G = S_A(G)$ and $R_A(G) = 0$, then the following are equivalent for an A-subsocle H of G:*

(a) *H is almost $\{A\}_*$-pure in G.*

(b) *If $U \subseteq G$ is A-projective of finite A-rank, then $(H + U)/H$ is torsion-free reduced.*

Proof. (a) ⇒ (b): By [3, Proposition 4.4], $H \oplus U = H \oplus C$ and $C \subseteq G \subseteq \prod_I A$ since $R_A(G) = 0$. Thus $(H + U)/H$ is torsion-free reduced.

(b) ⇒ (a): Let U be A-projective of finite A-rank in G. Consider the exact sequence

$$0 \to \operatorname{Hom}(A, H) \to \operatorname{Hom}(A, H) + \operatorname{Hom}(A, U) \to (\operatorname{Hom}(A, H) + \operatorname{Hom}(A, U))/\operatorname{Hom}(A, H) \to 0.$$

Suppose it does not split. Then there is an $E(A)$-submodule V of $\operatorname{Hom}(A, H) + \operatorname{Hom}(A, U)$ with $\operatorname{Hom}(A, H) + \operatorname{Hom}(A, U) = V \oplus W$ and $V/\operatorname{Hom}(A, H)$ is a finitely generated separable submodule of

$$(\operatorname{Hom}(A, H) + \operatorname{Hom}(A, U))/\operatorname{Hom}(A, H) \subseteq \operatorname{Hom}(A, (H + U)/H),$$

a torsion-free abelian group. Thus, $V/\operatorname{Hom}(A, H)$ is divisible. Moreover,

$$H + U = \theta_G(V \otimes_{E(A)} A) \oplus \theta_G(W \otimes_{E(A)} A)$$

and

$$H = \theta_G(\operatorname{Hom}(A, H) \otimes_{E(A)} A) \subseteq \theta_G(V \otimes_{E(A)} A).$$

Therefore

$$(H + U)/H \supseteq \theta_G(V \otimes_{E(A)} A)/H \cong (V/\operatorname{Hom}(A, H)) \otimes_{E(A)} A$$

is a divisible group, a contradiction. Thus, $\operatorname{Hom}(A, H) = V$. But then the top row of the following diagram splits:

$$\begin{array}{ccccc} 0 \to & \operatorname{Hom}(A, H) \otimes_{E(A)} A & \to & (\operatorname{Hom}(A, H) + \operatorname{Hom}(A, U)) \otimes_{E(A)} A \\ & \wr \downarrow \theta_G & & \wr \theta_G \\ 0 \to & H & \to & H + U \end{array}$$

Consequently, H is a direct summand of $H + U$.

We easily derive the required condition:

Corollary 6.6. *Let A be a torsion-free reduced abelian group with $E(A)$ right and left Noetherian, hereditary. The following are equivalent for an abelian group G:*

(a) *G is locally A-projective.*

(b) *$S_A(G) = G$, and there is an index-set I such that $G \subseteq \prod_I A$ and $(G + U)/G$ is torsion-free reduced for all A-projective groups U of finite A-rank in $\prod_I A$.*

COROLLARY 6.7. *Let A be as in Corollary* 6.6:

(i) *If P is a subgroup of $\oplus_n A$ with $S_A(P) = P$ and $(\oplus_n A)/P$ is torsion-free reduced, then P is a direct summand of $\oplus_n A$.*

(ii) (*Pontryagin's Lemma*) *An abelian group G is A-projective if $G = \bigcup_{n<\omega} B_n$, where $B_0 = 0$, $B_n \subseteq B_{n+1}$ are A-projective of finite A-rank and B_{n+1}/B_n is torsion-free reduced for all $n < \omega$.*

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